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Comments on Chiral Fermions in Stochastic Quantization

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Abstract. The subject of this Letter is twofold. First, a recently claimed failure of stochastic quantization scheme (SQS) to describe standard chiral anomalies at finite stochastic time is shown to result from an elementary mathematical error. The consistent approach of treating chiral fermions in SQS is briefly sketched and full agreement with previous investigations is established. Second, a serious new limitation on SQS is found, namely, its incapability to reproduce global chiral- and odd-dimensional parity-violating anomalies.

1. Generalized stochastic regularization (SR) [1] explicitly preserves chiral gauge symmetries in the stochastic quantization scheme (SQS) for chiral fermions. Therefore, it is an interesting test of SQS to check if the pertinent chiral anomalies are correctly reproduced after SR is removed.

In a recent note [2], a negative answer to this question was claimed. This failure of SQS, implying its incapability to describe massless fermions at finite fictitious stochastic time t, was attributed in [2] to the asserted noncommutativity of the zero mass limit ($m \to 0$) and the equilibrium limit ($t \to \infty$). These conclusions contradict previous considerations [3], where the recovery of the correct chiral anomalies within SQS (in a slightly different form from that in [2]) was explicitly demonstrated. The reason that anomalies appear within SQS is that the additional nonlocal terms introduced through SQS in the corresponding chiral Ward identities (cf. the r.h.s. of Equation (8) below) yield finite nonzero contributions (i.e., anomalies) when SR is removed (see Equations (9) and (10) below).

In Section 2 we briefly outline the SQS formulation for chiral (e.g., left-handed) fermions $\psi_L(t, x)$ interacting with a background U(n) gauge field $A_\mu(x)$ and the derivation of the correct standard (perturbative) chiral anomalies at finite t. Section 3 is devoted to the criticism of results and to the amendment of the approach of [2] so as to deduce the correct conclusions agreeing with those of [3] and of Section 2. On the other hand, some new serious drawbacks of SQS are pointed out in Section 4.

2. The Langevin equations for chiral fermions (in Euclidean D-dimensional spacetime) are taken in the form [3]:

$$\partial_t \psi_L^\perp = -(\mathscr{D}\mathscr{D}^*)\psi_L^\perp + \eta_L^\perp , \qquad \partial_t \overline{\psi}_L^\perp = -(\mathscr{D}^*\mathscr{D})^T \overline{\psi}_L^\perp + \overline{\eta}_L^\perp ; \tag{1}$$

$$\langle \eta_L^{\perp}(t,x)\bar{\eta}_L^{\perp}(t',x')\rangle = -2i\,\delta_\Lambda(t-t')\,\mathscr{D}(1-\Pi_0)\,\delta^{(D)}(x-x'), \tag{2}$$

with the following notations:

$$\begin{aligned}
&\mathscr{D}^{(*)} \equiv \mathscr{D}^{(*)}(A) = i\sigma_\mu^{(*)}\nabla_\mu(A) = i\sigma_\mu^{(*)}\,(\partial_\mu + iA_\mu(x)),\\
&\gamma_\mu = i\begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu^* & 0 \end{pmatrix}, \qquad \gamma^{(D+1)} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},\\
&\nabla\!\!\!/(A) = \gamma_\mu\nabla_\mu(A) = \begin{pmatrix} 0 & \mathscr{D}(A) \\ \mathscr{D}^*(A) & 0 \end{pmatrix},
\end{aligned} \tag{3}$$

$$\psi_L = \tfrac{1}{2}(\mathbb{1} + \gamma^{(D+1)})\psi, \qquad A_\mu(x) = A_\mu^a(x)T^a, \quad a = 0, 1, \ldots, n^2 - 1;$$

$$\psi_L^{\perp} \equiv (1 - \bar{\Pi}_0)\psi_L, \; \bar{\psi}_L^{\perp} \equiv (1 - \Pi_0)\bar{\psi}_L \quad \text{and analogously for } \overset{(-)}{\eta}{}_L^{\perp}. \tag{4}$$

The superscript 'T' in (1) means operator transposition, Π_0, $\bar{\Pi}_0$ in (2) and (4) are the zero-mode projectors of $\mathscr{D}^*\mathscr{D}$, $\mathscr{D}\mathscr{D}^*$, respectively. The Gaussian random source η_L in (1) is an anticommuting chiral spinor field with a two-point correlation function (2). SR is represented through the introduction of $\delta_\Lambda(t-t')$ in (2) obeying the following properties

$$\begin{aligned}
&\lim_{\Lambda\to\infty} \delta_\Lambda(t) = \delta(t), \qquad \delta_\Lambda(-t) = \delta_\Lambda(t),\\
&\frac{\mathrm{d}^k}{\mathrm{d}t^k}\,\delta_\Lambda(t)|_{t=0} = 0, \quad k = 0, 1, \ldots, L-1,
\end{aligned} \tag{5}$$

where L is an appropriate integer. A particular choice of (5) is

$$\delta_\Lambda(t) = \tfrac{1}{2}(L!)^{-1}\Lambda(\Lambda|t|)^L \exp\{-\Lambda|t|\}. \tag{5'}$$

The projection (4) in (1) and (2) is needed to guarantee the approach to equilibrium ($\mathscr{D}\mathscr{D}^*$ and $\mathscr{D}^*\mathscr{D}$ in (1) should be positive in order to yield a 'drift force'):

$$\langle F[\overset{(-)}{\psi}{}_L^{\perp}(x)]\rangle_Q = \lim_{t\to\infty} \langle F[\overset{(-)}{\psi}{}_L^{\perp}(t,x)]\rangle_\eta. \tag{6}$$

The subscript 'Q' on the l.h.s. of (6) denotes the usual Euclidean quantum average of an arbitrary functional $F[\overset{(-)}{\psi}{}_L^{\perp}(x)]$ with regularized standard weight $\exp\{-S_\Lambda\}$:

$$\begin{aligned}
&S_\Lambda = i\int \mathrm{d}^D x\, \mathrm{d}^D x'\, \bar{\psi}_L^{\perp}(x) K_\Lambda(x,x'|A_\mu)\psi_L^{\perp}(x'),\\
&\lim_{\Lambda\to\infty} K_\Lambda = \mathscr{D}^*,\\
&K_\Lambda^{-1}(x,x'|A_\mu) \equiv \lim_{t\to\infty} i\langle \psi_L^{\perp}(t,x)\bar{\psi}_L^{\perp}(t,x')\rangle_\eta\\
&\qquad = \mathscr{D}(\mathscr{D}^*\mathscr{D})^{-1}\left\{2\int_0^\infty \mathrm{d}\tau\,\delta_\Lambda(\tau)\exp\{-\tau\mathscr{D}^*\mathscr{D}\}(1-\Pi_0)\right\}(x,x')\\
&\qquad = \mathscr{D}(\mathscr{D}^*\mathscr{D})^{-1}\left[\frac{\Lambda}{\Lambda + \mathscr{D}^*\mathscr{D}}\right]^{L+1}(1-\Pi_0)(x,x')
\end{aligned} \tag{7}$$

(for the choice (5′)).

The subscript 'η' on the r.h.s. of (6) indicates the SQS average according to (2) with $\overset{(-)}{\psi}{}_L^{\perp}(t, x)$ being solutions of (1) subject to arbitrary initial conditions.

Choosing, e.g., $\psi_L^{\perp}(t = 0, x) = 0$ (in [3] another choice was made, $\psi_L^{\perp}(t = -\infty, x) = 0$), one gets in analogy with [3]:

$$
\begin{aligned}
&\nabla_\mu^{ab} J_\mu^{L,b}(t, x) \\
&\quad \equiv \nabla_\mu^{ab} \langle \overline{\psi}{}_L^{\perp}(t, x) T^b \sigma_\mu^* \psi_L^{\perp}(t, x) \rangle_\eta \\
&\quad = i \langle \{ [\mathscr{D}^* (\mathscr{D}\mathscr{D}^*)^{-1}]^T (\overline{\eta}{}_L^{\perp} - \partial_t \overline{\psi}{}_L^{\perp}) \} (t, x) \psi_L^{\perp}(t, x) \rangle_\eta - \\
&\quad\quad - \langle \overline{\psi}{}_L^{\perp}(t, x) \{ [\mathscr{D}^* (\mathscr{D}\mathscr{D}^*)^{-1}] (\eta_L^{\perp} - \partial t \psi_L^{\perp}) \} (t, x) \rangle_\eta \qquad (8) \\
&\quad = \lim_{\Lambda \to \infty} 2 \int_0^\infty \mathrm{d}\tau\, \delta_\Lambda(\tau) \theta(t - \tau) \operatorname{tr} \{ T^a [(\exp\{ -\tau \mathscr{D}^* \mathscr{D} \} - \\
&\quad\quad - \exp\{ -(2t - \tau) \mathscr{D}^* \mathscr{D} \})(1 - \Pi_0)(x, x) - \\
&\quad\quad - (\exp\{ -\tau \mathscr{D}\mathscr{D}^* \} - \exp\{ -(2t - \tau) \mathscr{D}\mathscr{D}^* \})(1 - \overline{\Pi}_0)(x, x)] \} \qquad (9) \\
&\quad = \lim_{\Lambda \to \infty} \operatorname{tr} \left[\left(\frac{\Lambda}{\Lambda + \mathscr{D}^* \mathscr{D}} \right)^{L+1} (1 - \Pi_0) - \left(\frac{\Lambda}{\Lambda + \mathscr{D}\mathscr{D}^*} \right)^{L+1} (1 - \overline{\Pi}_0) \right] (x, x) + O(\mathrm{e}^{-t})
\end{aligned}
$$

(for the choice (5')).

Now from (9) by a straightforward computation using, e.g., Seeley expansion of the heat kernels $\exp\{-\tau\mathscr{D}^*\mathscr{D}\}$, $\exp\{-\tau\mathscr{D}\mathscr{D}^*\}$ (e.g., [4]), one easily gets the standard covariant form of the (non-Abelian) chiral anomaly plus a correction for finite t vanishing in the equilibrium limit

$$
\begin{aligned}
\nabla_\mu^{ab} J_\mu^{L,b}(t, x) &= \operatorname{tr}[T^a(\overline{\Pi}_0 - \Pi_0)](x, x) - \mathscr{A}^a(x) + \\
&\quad + \operatorname{tr}\{ T^a [\exp\{ -2t \mathscr{D}\mathscr{D}^* \}(1 - \overline{\Pi}_0) - \exp\{ -2t \mathscr{D}^* \mathscr{D} \}(1 - \Pi_0)](x, x) \}; \\
\mathscr{A}^a(x) &\equiv \left[\left(\frac{D}{2} \right)! (4\pi)^{D/2} \right]^{-1} \varepsilon_{\mu_1 \ldots \mu_D} \operatorname{tr}[T^a F_{\mu_1\mu_2} \ldots F_{\mu_{D-1}\mu_D}]. \qquad (10)
\end{aligned}
$$

Let us particularly stress that SQS with SR (1), (2) manifestly preserves chiral gauge symmetries in the SQS averages, the r.h.s. of (6), for any t. However, this last property is *not* in conflict with the property of SQS with SR to correctly reproduce the standard chiral anomalies (9), (10). Indeed, on the one hand chiral anomalies result from the nonexistence of a consistent chiral gauge invariant definition of the chiral fermion effective action [5] which, in our case, reads (cf. (7))

$$S_\Lambda^{\mathrm{eff}} = -\ln \det[-iK_\Lambda] \qquad (11)$$

(recall that $K_\Lambda(7)$ is not an operator in one and the same space, but maps left-handed into right-handed spinors and, therefore, there is no unambiguous definition of $\det[-iK_\Lambda]$). On the other hand, the object S_Λ^{eff} (11) *cannot* be obtained within SQS as an appropriate SQS average according to (6). Hence, there is no clash between the chiral gauge noninvariance of (11) and the manifest chiral gauge symmetry of the stochastic averages (6).

3. In [2] the following SQS Langevin equations were chosen:

$$\begin{aligned}
&\partial_t \psi = -[\nabla\!\!\!/^2(A) + m^2]\psi + (-i\nabla\!\!\!/(A) + m)\eta_1 + \eta_2 ,\\
&\partial_t \overline{\psi} = -[\nabla\!\!\!/^2(A) + m^2]^T\overline{\psi} + (-i\nabla\!\!\!/(A) + m)^T\overline{\eta}_1 + \overline{\eta}_2 ;\\
&\langle \eta_a(t,x)\overline{\eta}_b(t',x')\rangle = \delta_{ab}\,\delta_\Lambda(t-t')\,\delta^{(D)}(x-x'), \quad a,b = 1,2,
\end{aligned} \tag{12}$$

where notations (3) are employed. Using (12) with initial conditions $\psi(t=0,x)=0$, one gets, in complete analogy with [3] and Section 2:

$$\begin{aligned}
\partial_\mu J_\mu^{(D+1)}(t,x) &\equiv \partial_\mu \langle \overline{\psi}(t,x)(-i\gamma_\mu)\gamma^{(D+1)}\psi(t,x)\rangle_\eta \\
&= \lim_{\Lambda\to\infty}(-4)\int_0^\infty \mathrm{d}\tau\,\delta_\Lambda(\tau)\theta(t-\tau)\,\mathrm{tr}\left\{\gamma^{(D+1)}\left[\left(1-\frac{m}{m+i\nabla\!\!\!/}\right)\times\right.\right.\\
&\quad\left.\left.\times\left(\exp\{-\tau(\nabla\!\!\!/^2+m^2)\} - \exp\{-(2t-\tau)(\nabla\!\!\!/^2+m^2)\}\right)\right](x,x)\right\}.
\end{aligned} \tag{13}$$

Equation (13) is an improved version of Equation (8) in [2] since here only mathematically well-defined objects (heat-kernels of elliptic positive operators $(m^2+\nabla\!\!\!/^2)$) are used instead of formal sums over eigenvalues of $\nabla\!\!\!/$ (whose spectrum is, in fact, continuous in general) as in [2]. We stress that in obtaining (9), (13) *no* additional regularization ('proper-time' cut-off, Pauli–Villars, etc.) besides SR was needed. After removing the SR in (13) and taking t large enough (to simplify the subsequent expressions) one finds:

$$\begin{aligned}
&\partial_\mu J_\mu^{(D+1)}(t,x)\\
&\quad = -2\mathscr{A}^0(x) + 2\,\mathrm{tr}[\gamma^{(D+1)}\exp\{-2t(\nabla\!\!\!/^2+m^2)\}(x,x)] + \\
&\quad + 2m^2\int_0^\infty \mathrm{d}\tau\,\mathrm{tr}\{\gamma^{(D+1)}[\exp\{-\tau(\nabla\!\!\!/^2+m^2)\} - \exp\{-(2t+\tau)(\nabla\!\!\!/^2+m^2)\}](x,x)\}.
\end{aligned} \tag{14}$$

Now it is evident from (14) that, contrary to the claim in [2], *no* difficulties do exist with the order of the limits $m\to 0$ and $t\to\infty$. Indeed, let us take, for instance, first $m\to 0$ with $t<\infty$ in (14) (this limit was claimed in [2] to annihilate the axial anomaly)

$$\begin{aligned}
\partial_\mu J_\mu^{(D+1)}(t,x) &= -2\mathscr{A}^0(x) + 2\,\mathrm{tr}[\gamma^{(D+1)}\Pi_0^{\nabla\!\!\!/}(x,x)] + \\
&\quad + 2\,\mathrm{tr}[\gamma^{(D+1)}(\exp\{-2t\nabla\!\!\!/^2\} - \Pi_0^{\nabla\!\!\!/})(x,x)],
\end{aligned} \tag{15}$$

where $\Pi_0^{\nabla\!\!\!/}(x,x')$ denotes the kernel of the zero-mode projector of $\nabla\!\!\!/$. Equation (15) is the correct version of the mathematically senseless Equation (12) of [2]

$$\begin{aligned}
&\partial_\mu J_\mu^{(D+1)}(t,x)\\
&\quad = -2\lim_{m\to 0}\sum_E \psi_E^*(x)\gamma^{(D+1)}\psi_E(x)[1-\exp\{-2t(E^2+m^2)\}]
\end{aligned} \tag{*}$$

($\nabla\!\!\!/\, \psi_E(x) = E\psi_E(x)$). Rewriting the r.h.s. of this last equation in terms of operator kernels

$$-2 \lim_{m\to 0} \mathrm{tr}[\gamma^{(D+1)}(\delta^{(D)}(0) - \exp\{-2t(\nabla\!\!\!/^2 + m^2)\})(x, x)]$$

one clearly sees that it is ill-defined.

The first two terms on the r.h.s. of (15) are immediately recognized as the standard nonintegrated axial anomaly, whereas the third term represents a correction for finite t which vanishes for $t \to \infty$. Upon integration of (15) over x, the last term gives zero for any t and, therefore, the standard index theorem $n_+ - n_- = \int \mathrm{d}^D x \mathscr{A}^0(x)$ is recovered for *any* finite t ($n_\pm$ denoting the numbers of chiral zero modes).

Thus, we have shown that the variant of SQS (12) for massless Dirac fermions chosen in [2], when carefully treated, also correctly reproduces the standard chiral anomalies★.

4. Finally, let us briefly demonstrate that SQS does in fact exhibit some new difficulties, but these occur for *global* (nonperturbative) chiral anomalies [6] as well as for massless fermions in *odd D*.

Indeed, according to [6] under homotopically nontrivial gauge transformations $\mathrm{U}(x) \in \mathrm{SU}(2)$ in $D = 4$:

$$\det[-i\nabla\!\!\!/(A)]^{1/2} = \det[-i\mathscr{D}^*(A)] = -\det[-i\mathscr{D}^*(A^U)]. \tag{16}$$

On the other hand, for each SU(2) gauge invariant quantity $\mathscr{F}[A_\mu]$ SQS gives

$$\langle \mathscr{F}[A_\mu(x)]\rangle = \left\{\int \mathscr{D}A_\mu \exp(-S_{\mathrm{eff}}[A])\right\}^{-1} \int \mathscr{D}A_\mu \exp\{-S_{\mathrm{eff}}[A]\}\mathscr{F}[A_\mu] \tag{17a}$$

$$= \lim_{t\to\infty} \langle \mathscr{F}[A_\mu(t, x)]\rangle_\eta, \tag{17b}$$

where $A_\mu(t, x)$ satisfies the Langevin equation

$$\partial_t A^a_\mu = -\delta S_{\mathrm{eff}}/\delta A^a_\mu + \eta^a_\mu, \tag{18}$$

$$S_{\mathrm{eff}}[A] = S_{\mathrm{YM}}[A] - \ln\det[-i\mathscr{D}^*(A)]. \tag{19}$$

Clearly, due to (16), i.e., the gauge noninvariance of S_{eff} (19), the r.h.s. of (17a) acquires the well-known uncertainty 0/0 [6], whereas (18) *is* gauge covariant and, therefore, the SQS average (17b) does not detect the global chiral anomaly (16) at finite stochastic time (further details will appear in a next note [7]).

Next, as discussed in the second reference in [3] SQS yields both a gauge- and parity-covariant expression for the induced massless fermion current in *odd D*:

★ Let us note that, even starting from the formal expression (*), upon integration over x one gets

$$0 = \int \mathrm{d}^D x\, \partial_\mu J^{(D+1)}_\mu(t, x) = -2 \lim_{m\to 0}(n_+ - n_-)(1 - \exp\{-2tm^2\}) = 0 \tag{**}$$

for $t < \infty$. Equation (**) should replace the incorrect Equation (13) of [2] where on the l.h.s., instead of zero as in (**), the authors put $\int \mathrm{d}^D x\, \partial_\mu J^{(D+1)}_\mu(t, x) = n_+ - n_-$??. Equation (**) is, of course, a trivial identity which does not contradict the index theorem, i.e., the integrated anomaly.

$$J_\mu^a(x) = \int_0^\infty \mathrm{d}\tau \, \mathrm{tr}[T^a \gamma_\mu \not\nabla(A) \exp\{-\tau \not\nabla^2(A)\}(x, x)] = (2i)^{-1} \frac{\delta}{\delta A_\mu^a}(x) \ln \det[\not\nabla^2(A)]$$

Hence, SQS fails to reproduce the corresponding parity-violating anomalies in odd D [8] (when the latter cannot be cancelled by appropriate counterterms [9])

$$J_\mu^a(x) = -i \frac{\delta}{\delta A_\mu^a}(x) \left\{ \tfrac{1}{2} \ln \det[\not\nabla^2(A)] - i \frac{\pi}{2} \eta_{\not\nabla}[A] \right\},$$

where $\eta_{\not\nabla}[A]$ is the parity-odd spectral asymmetry measuring η-invariant [10] of $\not\nabla(A)$. This fact poses serious restrictions on the applicability of SQS to fermions in odd D.

The reason for the failure just found of SQS to reproduce anomalies of discrete symmetries is easy to understand. SR preserves these discrete symmetries. However, unlike the case of anomalies of continuous symmetries, there are no modified by SQS Ward identities for the corresponding discrete symmetries, so that the mechanism for a raising of the standard chiral anomalies, described in Section 2, does not operate here.

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